Generating new solutions for relativistic transverse flow at the softest point

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Abstract: Using the method of prolongation we generate new stationary solutions from a recently obtained simple particular solution for relativistic transverse flow with cylindrical symmetry in 1+3 dimension. This is an extension of the longitudinal Bjorken flow ansatz and can be applied to situations during a first order phase transition in a fast expanding system. The prolongated solution allows us to trace back the flow profile from any transverse flow conjectured at the end of the phase transition.

Introduction

Hydrodynamics often allow for nonrelativistic scaling solutions. Relativistic flow, however, seemed long to be an exception: besides Bjorken's 1+1 dimensional ansatz and the spherically symmetric relativistic expansion, no analytical solution was known [2].

In a recent paper [1] we presented an extension of Bjorken's ansatz [3] for longitudinally and transversally relativistic flow patterns with cylindrical symmetry in 1+3 dimensions. It satisfies $u^\mu \partial_\mu u^\nu = 0$ as well as the original Bjorken ansatz. This analytical solution of the flow equations of a perfect fluid is in particular valid for physical situations when the sound velocity is zero, $c_s^2 = dp/d\epsilon = 0$, with energy density ϵ and pressure $p(\epsilon)$.

In particular this happens during a first order phase transition, the pressure is constant while the energy density changes (in heavy ion collisons increases and drops again). This should, in principle, be signalled by a vanishing sound velocity. A remnant of this effect in finite size, finite time transitions might be a softest point of the equation of state, where c_s^2 is minimal. In fact, this has been suggested as a signal of phase transition by Shuryak[4], and investigated numerically in several recent works [5, 6].

In this paper we generalize that simple analytic solution further by exploring the symmetries of the nonlinear partial differential equation determining the flow at the softest point (the relativistic Euler's equation). The method of prolongation ensures us that all possible transformations of dependent and independent variables can be found which comply with the equations to be solved by solving linear partial differential equations only. Then

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from any simple (or trivial) particular solution of the original nonlinear problem several new classes of solutions can be generated. In this article we present the simple solution published recently, we review the method of prolongation and obtain a general class of transverse flow patterns for the relativistic problem at the softest point. We use this solution for calculating backwards from a conjectured final state at CERN SPS [8].

Nonrelativistic analytic solution has been also given several times, with respect to heavy ions see [9, 10, 11].

Simple relativistic transverse flow at the softest point

The 1+1 dimensional Bjorken flow four-velocity is a normalized, timelike vector. It is natural to choose this as the first of our comoving frame basis vectors (vierbein). The three further, spacelike vectors will be constructed orthogonal to this, separating the two transverse directions. This basis fits excellently to a cylindrical symmetry and to longitudinally extreme relativistic flow.

$$\begin{aligned} e^0_\mu &= \left(\frac{t}{\tau}, \frac{z}{\tau}, 0, 0\right), & e^1_\mu &= \left(\frac{z}{\tau}, \frac{t}{\tau}, 0, 0\right), \\ e^2_\mu &= \left(0, 0, \frac{x}{r}, \frac{y}{r}, \right), & e^3_\mu &= \left(0, 0, -\frac{y}{r}, \frac{x}{r}, \right), \end{aligned} \tag{1}$$

with t time coordinate and z longitudinal (beam-along) coordinate, x and y transverse, cartesian coordinates. The cylindrical radius is given by, $r = \sqrt{x^2 + y^2}$, and τ is the *longitudinal proper time*: $\tau = \sqrt{t^2 - z^2}$.

We consider ideal fluids ("dry water"), where the energy momentum tensor is given by

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{2}$$

and the equation of state is given in the form of $p(\epsilon)$. The ansatz for an almost boost invariant flow with some transverse, cylindrically symmetric component is then given by

$$u_\mu = \gamma \left(e^0_\mu + v e^2_\mu\right), \tag{3}$$

using the Lorentz factor $\gamma = (1 - v^2)^{-1/2}$. This four-velocity is normalized to one:

$$u_\mu u^\mu = \gamma^2 - \gamma^2 v^2 = 1. \tag{4}$$

In a recent publication we formulated Euler's equation in terms of this ansatz and coordinates. For a situation with zero comoving gradient of the pressure it has been simplified to a sole partial differential equation for the transverse flow velocity component $v(\tau, r)$:

$$\left(\frac{\partial}{\partial \tau} + v \frac{\partial}{\partial r}\right) v = 0. \tag{5}$$

This equation is valid even for relativistic transverse flow, it is a quasilinear partial differential equation. The separable solution,$v(r,\tau) = a(r)b(\tau)$, which is regular at the cylindrical axis $r=0$ is given by

$$v = \frac{r}{\tau}. \tag{6}$$

In a recent paper we then explored consequences of such a linear transverse velocity profile for the time evolution of the energy density at constant pressure, i.e. the quark matter to hadron matter phase conversion[1].

Our concern now is to construct a more general solution of eq.(5), which is able to fit *any* flow profile at a given longitudinal time

$$v(\tau_0, r) = v_0(r). \tag{7}$$

In order to explore the quark matter hadron matter phase conversion we utilize simple bag model equation of state, as was done in [1]. The energy density of χ part quark matter and $(1-\chi)$ hadron matter is then given by

$$e = \left(\sigma_q T_c^4 + B\right)\chi + \sigma_h T_c^4 (1-\chi), \tag{8}$$

at the phase equilibrium temperature T_c, determined from the equity of quark and hadron pressure. The resulting evolution equation turns to be

$$\partial_\mu(\chi u^\mu) + \nu \partial_\mu u^\mu = 0, \tag{9}$$

with

$$\nu = \frac{4}{3}\frac{\sigma_h}{\sigma_q - \sigma_h}. \tag{10}$$

Here the ratio of Stefan-Boltzmann constants $\sigma_{q,h}$ depends only on the relative number of the effective degrees of freedom. By comparing a relativistic pion gas to quark gluon plasma $\sigma_h = 3$ and $\sigma_q = 37$ and one obtains $\nu \approx 0.12$.

Now using the flow ansatz discussed above and eq.(6) one arrives at

$$\left(\frac{\partial}{\partial \tau} + v\frac{\partial}{\partial r}\right)\chi + (\chi + \nu)\left(\frac{\partial v}{\partial r} + \frac{v}{r} + \frac{1}{\tau}\right) = 0. \tag{11}$$

Furthermore, expressing the quark matter part as

$$\chi(\tau, r) = -\nu + \frac{1}{\tau r} Z(\tau, r), \tag{12}$$

we obtain a simpler equation for Z,

$$\frac{\partial}{\partial \tau}Z + \frac{\partial}{\partial r}(vZ) = 0, \tag{13}$$

which formally resembles a one-dimensional conservation equation with density Z and convective current vZ. It is noteworthy that for any solution Z of this equation (13)

$f(v)Z$ is also a solution with $f(v)$ being an arbitrary (but differentiable) function of the transverse velocity v. This freedom may be used for fitting a quark matter percentage profile at the same time when the transverse velocity profile is known, especially the $\chi = 0$ (end of phase conversion) situation may be traced back up to a time when somewhere $\chi = 1$ (full quark matter) is reached.

Eq.(13) is formally a conservation equation, whose solution, $Z(\tau, r)$ satisfies

$$Q(\tau) = \int_0^{R(\tau)} Z(\tau, r)\, dr = Q(\tau_0). \tag{14}$$

The longitudinal time derivative of this expression vanishes leading to

$$\frac{dQ}{d\tau} = \dot{R}(\tau) Z(\tau, r) + \int_0^{R(\tau)} \frac{\partial Z}{\partial \tau}\, dr = 0. \tag{15}$$

Replacing $\partial Z / \partial \tau$ from eq.(13) we arrive at

$$\left[\dot{R}(\tau) - v(\tau, R(\tau)) \right] Z(\tau, R(\tau)) = 0 \tag{16}$$

if we take into account the regularity condition $v(\tau, 0) = 0$. This is an evolution equation for the transverse radius $R(\tau)$ of the phase mixture region where $Z \neq 0$,

$$\dot{R}(\tau) \; = \; v(\tau, R(\tau)). \tag{17}$$

It is a first order ordinary differential equation which can be easily solved once $v(\tau, r)$ is known.

The conserved quantity $Q(\tau)$ can be related to the spatial average of the phase mixture ratio by using eq.(12):

$$\langle \chi \rangle = \frac{2}{R^2} \int_0^R \chi(\tau, r) r dr. \tag{18}$$

Finally we obtain

$$\langle \chi \rangle = -\nu + \frac{2}{\tau R^2(\tau)} Q(\tau). \tag{19}$$

In the followings we turn to a brief presentation of the method of prolongation, which shall be used for generalizing the particular relativistic transverse flow solution presented in [1].

The method of prolongation

The method of prolongation "prolonges" symmetries of a (system of) nonlinear partial differential equations, to transformations of dependent and independent variables and all partial derivatives of the dependent variables up to a degree less then the rank of the equation itself. For an incomplete list of references about the application of this method in physics see [12].

Let us generally denote the set of independent variables by x, the dependent ones by v, partial derivatives by v_x, v_{xx} etc. The equation(s) to be solved let be given by

$$\Delta(x, v, v_x, \ldots) = 0. \tag{20}$$

Transformations on x and v (generating corresponding transformations on the partial derivatives) may let this equation remain valid in the transformed variables as well. These transformations are symmetries of the equation, and they can be viewed as integrals of infinitesimal transformations generated by the vector field (Killing vectors)

$$K = \Phi(u, x)\frac{\partial}{\partial v} + \xi(v, x)\frac{\partial}{\partial x}. \tag{21}$$

This relation can be easily understood by inspecting infinitesimal transformations,

$$\begin{aligned} \tilde{v} &= v + \varepsilon\Phi, \\ \tilde{x} &= x + \varepsilon\xi. \end{aligned} \tag{22}$$

In this case the derivative transforms to

$$\frac{d\tilde{v}}{d\tilde{x}} = \frac{\frac{dv}{dx} + \varepsilon\frac{d\Phi}{dx}}{1 + \varepsilon\frac{d\xi}{dx}} = \frac{dv}{dx} + \varepsilon\left(\frac{d\Phi}{dx} - \frac{d\xi}{dx}\frac{dv}{dx}\right) + \mathcal{O}(\varepsilon^2) \tag{23}$$

The total derivatives are

$$\begin{aligned} \frac{d\Phi}{dx} &= \Phi_v v_x + \Phi_x, \\ \frac{d\xi}{dx} &= \xi_v v_x + \xi_x. \end{aligned} \tag{24}$$

The differential equation, $\Delta = 0$ can be expanded around the original solution

$$\Delta(\tilde{v}, \tilde{x}, \tilde{v}_x) = \Delta(v, x, v_x) + \varepsilon\Phi\frac{\partial}{\partial v}\Delta + \varepsilon\xi\frac{\partial}{\partial x}\Delta + \varepsilon\left(\frac{d\Phi}{dx} - \frac{d\xi}{dx}v_x\right)\frac{\partial}{\partial v_x}\Delta + \mathcal{O}(\varepsilon^2). \tag{25}$$

The second and third term of $\mathcal{O}(\varepsilon)$ in the above expansion resembles $\hat{K}\Delta$, but all terms to this order constitute a Killing field in the extended (prolongated) space spanned by v, x and v_x. This is the (first) prolongation of the symmetry

$$\mathrm{pr}^{(1)}K = K + \Phi^x\frac{\partial}{\partial v_x} \tag{26}$$

with

$$\Phi^x = \frac{d\Phi}{dx} - v_x \frac{d\xi}{dx} = \frac{d}{dx}(\Phi - v_x\xi) + \xi v_{xx}. \tag{27}$$

Since both $v(x)$ and $\tilde{v}(\tilde{x})$ are solutions, $\Delta(\tilde{v}, \tilde{x}, \tilde{v}_x) = 0$ and $\Delta(v, x, v_x) = 0$ for arbitrary ε and we arrive at

$$\mathrm{pr}^{(1)} K(\Delta)\Big|_{\Delta=0} \;=\; 0. \tag{28}$$

The requirement of transforming only among solutions leads to a number of *linear* partial differential equations for the unknown functions Φ and ξ (which may contain, in general, several components) by equating the coefficients of each monomials in v, v_x etc. with zero. Eq.(28) is a property of the transformation and therefore is fulfilled for any solution of the original equation. The $\Delta = 0$ constraint is important, it reduces the dimensionality of the space of the coefficient functions, usually by expressing one of the highest oder partial derivatives in terms of others. The resulting system of equations in principle can be solved (being linear). The corresponding vector field K will then contain in general a number of constants or undetermined functions selecting out classes of transformations which generate new solutions from a given particular solution (20). The method of prolongation ensures us that we have considered all possible transformations.

New solutions

Now we rewrite equation (5) with notations of the method of prolongation:

$$\begin{aligned} \Delta &= v_\tau + v v_r = 0, \\ K &= \Phi(v,\tau,r)\frac{\partial}{\partial v} + T(v,\tau,r)\frac{\partial}{\partial \tau} + R(v,\tau,r)\frac{\partial}{\partial r}, \\ \mathrm{pr}^{(1)} K &= K + \Phi^\tau \frac{\partial}{\partial v_\tau} + \Phi^r \frac{\partial}{\partial v_r}. \end{aligned} \tag{29}$$

Applying the prolonged Killing vector field to the nonlinear equation to be solved we get

$$\mathrm{pr}^{(1)} K\Delta = \Phi^\tau + v\Phi^r + v_r\Phi = 0. \tag{30}$$

The prolonged coefficients of the Killing vector field are

$$\begin{aligned} \Phi^\tau &= (\Phi_\tau + v_\tau\Phi_v) - (T_\tau v_\tau + T_v v_\tau^2) - (R_\tau v_r + R_v v_\tau u_r), \\ \Phi^r &= (\Phi_r + v_r\Phi_v) - (T_r v_\tau + T_v v_\tau u_r) - (R_r v_r + R_v v_r^2). \end{aligned} \tag{31}$$

Replacing it to the prolongation condition eq.(30), we arrive at

$$\mathrm{pr}^{(1)} K\;(\Delta)|_{\Delta=0} \;=\; \Phi_\tau + v\Phi_r + v_r(\Phi - R_\tau) + v v_r(T_\tau - R_r) + v^2 v_r T_r = 0. \tag{32}$$

This has to be fulfilled for arbitrary $v(\tau, r)$. In this equation, since we have already used the $\Delta = 0$ constraint (the original equation to be solved) for eliminating $v_\tau = -vv_r$, each coefficient in the polynomial of v and v_r vanishes separately, leading to a number of *linear* partial differential equations:

$$\Phi_\tau = 0, \qquad \Phi_r = 0, \qquad \Phi = R_\tau, \qquad T_\tau = R_r, \qquad T_r = 0. \tag{33}$$

To these equations (being linear) a general solution can be given:

$$\begin{aligned} T(v, \tau, r) &= b(v) + a(v)\tau, \\ R(v, \tau, r) &= c(v) + a(v)r + \phi(v)\tau, \\ \Phi(v, \tau, r) &= \phi(v). \end{aligned} \tag{34}$$

There are four undetermined functions in this general solution to the Killing vector problem, hence there are four general classes of transformations leaving the fulfillment of the original equation untouched. The corresponding vector fileds, which generate infinitesimal transformations, are

$$\begin{aligned} K_1 &= \phi(v)\left(\frac{\partial}{\partial v} + \tau\frac{\partial}{\partial r}\right), \\ K_2 &= a(v)\left(r\frac{\partial}{\partial r} + \tau\frac{\partial}{\partial \tau}\right), \\ K_3 &= b(v)\frac{\partial}{\partial \tau}, \\ K_4 &= c(v)\frac{\partial}{\partial r}. \end{aligned} \tag{35}$$

A general finite transformation of any solution of $\Delta = 0$ is given by

$$(\tilde{v}, \tilde{\tau}, \tilde{r}) = \exp\left(\epsilon_1 K_1 + \epsilon_2 K_2 + \epsilon_3 K_3 + \epsilon_4 K_4\right)(v, \tau, r). \tag{36}$$

Three of these transformations are quite trivial: K_4 generates translation of the variable r by $c(v)$, K_3 generates translation of the variable τ by $b(v)$, and K_2 a strecth of both r and τ by $e^{a(v)}$. Only K_1 involves a transformation of v as well:

$$\begin{aligned} \tilde{v} &= v + \psi(v), \\ \tilde{\tau} &= \tau, \\ \tilde{r} &= r + \tau\psi(v), \end{aligned} \tag{37}$$

with

$$\psi(v) = \int \frac{dv}{\Phi(v)}. \tag{38}$$

Combination of these transformations generates from the simple particular solution $v = r/\tau$ the following (implicit) solution

$$v = \frac{r + A(v)}{\tau + B(v)}. \tag{39}$$

Fitting this form to an initial transverse velocity profile, leads to

$$v_0 = v(0, r) = \frac{r + A(v_0(r))}{B(v_0(r))}, \tag{40}$$

which is equivalent to

$$r = v_0 B(v_0) - A(v_0) = f(v_0). \tag{41}$$

Finally we arrive at the general solution

$$r - v\tau = f(v), \tag{42}$$

with $f(v)$ being the inverse function of the initial transverse velocity profile $v_0(r)$. The general solution is implicitely given by

$$v = v_0(r - v\tau). \tag{43}$$

It is easy to verify that this implicit form still satisfies the original equation (5).

Finally let us consider some possible initial transverse velocity profiles. For the linear case,

$$v_0(r) = \alpha r, \tag{44}$$

we obtain

$$v = \alpha(r - v\tau) \tag{45}$$

which can be resolved to the explicit expression

$$v = \frac{\alpha r}{1 + \alpha\tau} = \frac{r}{\tau + 1/\alpha}. \tag{46}$$

presented already in [1]. Initial profiles with nonlinear or non-invertable functions can be resolved only numerically, but a solution of implicit equations are usually faster and more stable than algorithms for solving partial differential equations numerically.

Flow at SPS

A particularly interesting enterprise is to start with a transverse flow profile conjectured from experimental data and evolve the flow pattern backwards in (longitudinal) time. For the CERN SPS heavy ion experiments Ster, Csörgő and Lörstad have proposed recently such a profile based on one-particle transverse momentum spectra and HBT correlation measurments [8]. The reconstructed flow in the notations of the present paper can be written as,

$$\gamma v = \sinh \eta_t = \langle u_t \rangle \frac{r_t}{R_G} = \alpha r, \tag{47}$$

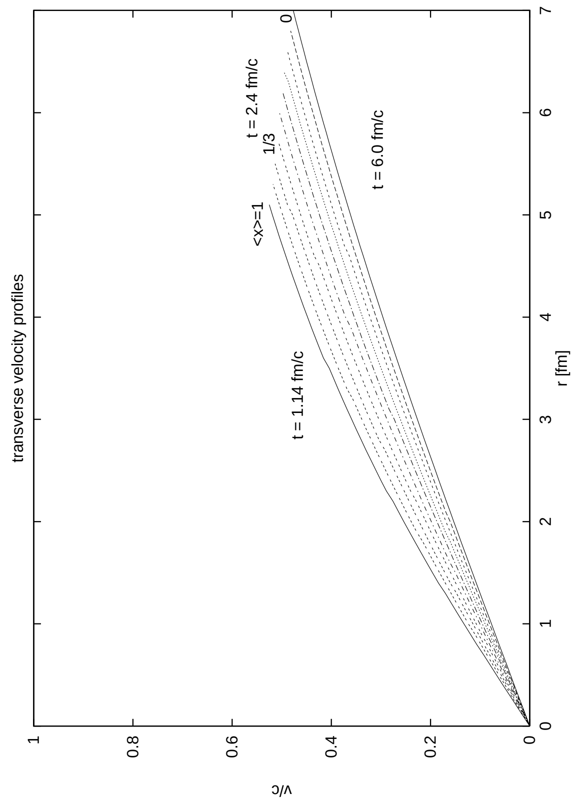

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Figure 1: Relativistic transverse velocity profiles from the prolonged analytic solution. The curves are results of a time backward calculation from a conjectured final state at $\tau = 6$ fm/c with a radius of $R = 7.1$ fm. The average portion of the quark matter in the mixed phase is indicated at the respective values of 0, 1/3 and 1.

at a late time instant $\tau_0 = 6$ fm/c. The fit by Ster et. al. concludes at $\langle u_t \rangle = 0.55 \pm 0.06$ and $R_G = 7.1 \pm 0.2$ fm. This gives rise to the following proportionality coefficient: $\alpha = 0.0775 \pm 0.01$. This suggestion has improved on the linear transverse flow assumption, it nowhere exceeds the speed of light.

Fig.1 shows the reconstructed relativistic transverse velocity profiles ending at the one suggested by Ster et.al. It was assumed that in the final stage the phase transition just ended in an overall hadronic phase.

As insertions show the mixed phase should have started then as early as $\tau = 1.14$ fm/c, but the quark matter part was decreased to 1/3 on the average soon ($\tau = 2.4$ fm/c). The velocity profiles do not differ from linear qualitatively in the range of interest. It is more pronounced how the radius of the mixed phase expands from $R_0 = 5.4$ fm to the reconstructed final stage with $R = 7.1$ fm.

Conclusion

With the help of the method of prolongation we explored the symmetries of Euler's equation for a relativistic transverse flow at the softest point (zero comoving gradient for pressure). We arrived at an implicit solution (43) which presents a correspondence between the initial transverse velocity profile and that at arbitrary longitudinal proper time. Although in a general case an implicit equation can only be solved numerically, it is a more elegant, stable and concise way to solve the flow problem than the numerical integration of partial differential equations.

With this perspective a transverse flow pattern obtained from measurements in heavy ion experiments at CERN SPS, where probably the mixed quark-gluon and hadronic phase has been realized, can also be re-calculated at the beginning of phase conversion (pure quark matter). In turn this might help for obtaining an improved insight into further characteristics (temperature, effective masses) of the quark matter at CERN.

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